

—Chapter 3—

Functional Field Integral

3-1 Fermion Coherent States

A. GRASSMANN ALGEBRA

- (1) The eigenstates of the (non-Hermitian) Fock space operators $\hat{c}_i$

$$\hat{c}_i|\psi\rangle = \psi_i|\psi\rangle$$

where ψ_i is the eigenvalue.

Since $\{\hat{c}_i, \hat{c}_j\} = 0$ with $i \neq j$

$$\{\hat{c}_i, \hat{c}_j\}|\psi\rangle = (\hat{c}_i\hat{c}_j + \hat{c}_j\hat{c}_i)|\psi\rangle = (\psi_i\psi_j + \psi_j\psi_i)|\psi\rangle = 0$$

$$\Rightarrow \psi_i\psi_j = -\psi_j\psi_i$$

Anticommuting eigenvalues are called Grassmann numbers which are not ordinary numbers.

- (2) Grassmann algebra: define a set of $2N$ anticommuting variables $\{\bar{\psi}_i, \psi_i\}, i = 1, 2, \dots, N$, which satisfy the following conditions:

$$\psi_i\psi_j + \psi_j\psi_i = 0$$

$$\bar{\psi}_i\psi_j + \psi_j\bar{\psi}_i = 0$$

$$\bar{\psi}_i\bar{\psi}_j + \bar{\psi}_j\bar{\psi}_i = 0$$

where ψ_i and $\bar{\psi}_i$ are complex conjugate.

OS:

Do not confuse Grassmann numbers with fermionic creation and annihilation operators.

Thus, we obtain $\psi_i^2 = \bar{\psi}_i^2 = 0$ and $(\psi_1 \cdots \psi_N)^* = \bar{\psi}_N \cdots \bar{\psi}_1$.

- (3) Functions of Grassmann variables:

1. One Grassmann variable ψ

$$f(\psi) = f_0 + f_1\psi + f_2\psi^2 + f_3\psi^3 + \cdots$$

Since $\psi^2 = 0$

$$f(\psi) = f_0 + f_1\psi$$

Similarly,

$$\overline{f(\psi)} = \bar{f}_0 + \bar{f}_1\bar{\psi}$$

2. Two Grassmann variables ψ and $\bar{\psi}$

$$A(\psi, \bar{\psi}) = a_0 + a_1\psi + a_2\bar{\psi} + a_{12}\bar{\psi}\psi$$

- (4) Differentiation: the derivative is identical to the ordinary derivative except that the variable has to be adjacent to the derivative operator,

$$\begin{aligned}\frac{\partial}{\partial\psi}(\bar{\psi}\psi) &= \frac{\partial}{\partial\psi}(-\psi\bar{\psi}) = -\bar{\psi} \\ \begin{cases} \frac{\partial}{\partial\psi}A(\psi, \bar{\psi}) = a_1 - a_{12}\bar{\psi} \\ \frac{\partial}{\partial\bar{\psi}}A(\psi, \bar{\psi}) = a_2 + a_{12}\psi \end{cases} \\ \Rightarrow \frac{\partial}{\partial\psi}\frac{\partial}{\partial\bar{\psi}}A(\psi, \bar{\psi}) &= a_{12} = -\frac{\partial}{\partial\bar{\psi}}\frac{\partial}{\partial\psi}A(\psi, \bar{\psi}) \\ \Rightarrow \frac{\partial}{\partial\psi}\frac{\partial}{\partial\bar{\psi}} + \frac{\partial}{\partial\bar{\psi}}\frac{\partial}{\partial\psi} &= 0 \\ \Rightarrow \text{Two derivative operators anticommute.}\end{aligned}$$

Integration: the Grassmann integral is a linear mapping and defined as the Grassmann derivative:

$$\int d\psi = \frac{\partial}{\partial\psi}$$

Thus, we have

$$\int d\psi \, 1 = 0, \quad \int d\psi \, \psi = 1$$

Integral of analytic functions

$$\begin{aligned}\int d\psi f(\psi) &= \int d\psi (f_0 + f_1\psi) = f_1 \\ \int d\psi A(\psi, \bar{\psi}) &= \int d\psi (a_0 + a_1\psi + a_2\bar{\psi} + a_{12}\bar{\psi}\psi) = a_1 - a_{12}\bar{\psi}\end{aligned}$$

Grassmann δ -function

$$\begin{aligned}\delta(\psi - \psi') &= \int d\eta e^{-\eta(\psi - \psi')} = \int d\eta (1 - \eta(\psi - \psi')) = -(\psi - \psi') \\ \int d\psi' \delta(\psi - \psi') f(\psi') &= - \int d\psi' (\psi - \psi')(f_0 + f_1\psi') \\ &= \int d\psi' (-f_0\psi + f_0\psi' - f_1\psi\psi' + f_1\psi'\psi') \\ &= 0 + f_0 + \int d\psi' f_1\psi'\psi + 0 \\ &= f_0 + f_1\psi \\ &= f(\psi)\end{aligned}$$

- (5) Gaussian integral over a pair of complex Grassmann variables

$$\int d\bar{\psi}d\psi e^{-a\bar{\psi}\psi} = \int d\bar{\psi}d\psi (1 - a\bar{\psi}\psi) = a$$

$$\int d\bar{\psi}d\psi \psi\bar{\psi}e^{-a\bar{\psi}\psi} = \int d\bar{\psi}d\psi \psi\bar{\psi}(1 - a\bar{\psi}\psi) = \int d\bar{\psi}d\psi \psi\bar{\psi} = 1$$

Gaussian integral over N -pair of complex Grassmann variables

1. $\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} = \det \mathbf{A}$
2. $\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j + \sum_i (\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i)} = (\det \mathbf{A}) e^{\sum_{ij} \bar{\eta}_i A_{ij}^{-1} \eta_j}$
3. $\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i \psi\bar{\psi} e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} = (\det \mathbf{A})(\mathbf{A}^{-1})$

where A_{ij} is an ordinary number and $\mathbf{A}$ is a Hermitian matrix.

PROOF:

1. Define a linear transformation

$$\phi_i = \sum_j A_{ij} \psi_j$$

$$\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} = \int \prod_{i=1}^N d\bar{\psi}_i d\phi_i J e^{-\sum_i \bar{\psi}_i \phi_i}$$

The Jacobian rule is

$$\int \prod_{i=1}^N d\psi_i \psi_i = 1 = \int \prod_{i=1}^N d\phi_i \phi_i = \int \prod_{i=1}^N d\phi_i \det \frac{\partial \phi_i}{\partial \psi_j} \psi_j$$

$$\prod_{i=1}^N d\psi_i = \det \mathbf{A} \prod_{i=1}^N d\phi_i, \quad \text{The Jacobian is } |\det \mathbf{A}|$$

NOTE:

This is opposite of the Jacobian rule for ordinary integrals.

$$\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} = \det \mathbf{A} \int \prod_{i=1}^N d\bar{\psi}_i d\phi_i e^{-\sum_i \bar{\psi}_i \phi_i}$$

$$= \det \mathbf{A} \prod_{i=1}^N \int d\bar{\psi}_i d\phi_i (1 - \bar{\psi}_i \phi_i)$$

$$= \det \mathbf{A}$$

Alternative approach:

Make a unitary similarity transformation to diagonalize $\mathbf{A}$

$$\mathbf{D} = \mathbf{U}^\dagger \mathbf{A} \mathbf{U}$$

$$\text{Let } \zeta_i = \sum_j U_{ij}^\dagger \psi_j, \bar{\zeta}_i = \sum_j \bar{\psi}_j U_{ji}$$

$$\sum_i \bar{\zeta}_i D_{ii} \zeta_i = \sum_{ikl} (\bar{\psi}_k U_{ki}) D_{ii} (U_{il}^\dagger \psi_l) = \sum_{ij} \bar{\psi}_i A_{ij} \psi_j$$

The Jacobian is $|\det \mathbf{U}| = 1$

$$\begin{aligned} \int \prod_{i=1}^N d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} &= \int \prod_{i=1}^N d\bar{\zeta}_i d\zeta_i e^{-\sum_i \bar{\zeta}_i D_{ii} \zeta_i} \\ &= \prod_{i=1}^N \int d\bar{\zeta}_i d\zeta_i (1 - \bar{\zeta}_i D_{ii} \zeta_i) \\ &= \prod_{i=1}^N D_{ii} = \det \mathbf{D} = \det \mathbf{A} \end{aligned}$$

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2. Define the transformations

$$\begin{aligned} \rho_i &= \psi_i - \sum_j A_{ij}^{-1} \eta_j \\ \bar{\rho}_i &= \bar{\psi}_i - \sum_j \bar{\eta}_j A_{ji}^{-1} \\ \sum_{ij} \bar{\rho}_i A_{ij} \rho_j &= \sum_{ij} \left(\bar{\psi}_i - \sum_k \bar{\eta}_k A_{ki}^{-1} \right) A_{ij} \left(\psi_j - \sum_l A_{jl}^{-1} \eta_l \right) \\ &= \sum_{ij} \bar{\psi}_i A_{ij} \psi_j - \sum_{ijk} \bar{\eta}_k A_{ki}^{-1} A_{ij} \psi_j - \sum_{ijl} \bar{\psi}_i A_{ij} A_{jl}^{-1} \eta_l \\ &\quad + \sum_{ijkl} \bar{\eta}_k A_{ki}^{-1} A_{ij} A_{jl}^{-1} \eta_l \\ &= \sum_{ij} \bar{\psi}_i A_{ij} \psi_j - \sum_{jk} \bar{\eta}_k \psi_j \delta_{kj} - \sum_{il} \bar{\psi}_i \eta_l \delta_{il} + \sum_{jkl} \bar{\eta}_k A_{jl}^{-1} \eta_l \delta_{kj} \\ &= \sum_{ij} \bar{\psi}_i A_{ij} \psi_j - \sum_i \bar{\eta}_i \psi_i - \sum_i \bar{\psi}_i \eta_i + \sum_{ij} \bar{\eta}_i A_{ij}^{-1} \eta_j \end{aligned}$$

Since the Jacobian rule is

$$\begin{aligned}
\int \prod_{i=1}^N d\psi_i \psi_i &= 1 = \int \prod_{i=1}^N d\rho_i \rho_i \\
\prod_{i=1}^N d\psi_i &= \prod_{i=1}^N d\rho_i, \quad \text{The Jacobian is 1} \\
\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j + \sum_i (\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i)} \\
&= \int \prod_{i=1}^N d\bar{\rho}_i d\rho_i e^{-\sum_{ij} \bar{\rho}_i A_{ij} \rho_j + \sum_{ij} \bar{\eta}_i A_{ij}^{-1} \eta_j} \\
&= e^{\sum_{ij} \bar{\eta}_i A_{ij}^{-1} \eta_j} \int \prod_{i=1}^N d\bar{\rho}_i d\rho_i e^{-\sum_{ij} \bar{\rho}_i A_{ij} \rho_j} \\
&= (\det \mathbf{A}) e^{\sum_{ij} \bar{\eta}_i A_{ij}^{-1} \eta_j}
\end{aligned}$$

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3. With the help of identities

$$\begin{aligned}
\bar{\psi}_i e^{\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i} &= -\frac{\partial}{\partial \eta_i} e^{\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i} \\
\psi_i e^{\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i} &= \frac{\partial}{\partial \bar{\eta}_i} e^{\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i \prod_{i=1}^N \psi_i \bar{\psi}_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j} \\
&= -\int \prod_{i=1}^N d\bar{\psi}_i d\psi_i \prod_{i=1}^N \frac{\partial^2}{\partial \bar{\eta}_i \partial \eta_i} e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j + \sum_i (\bar{\eta}_i \psi_i + \bar{\psi}_i \eta_i)} \\
&= -\prod_{i=1}^N \frac{\partial^2}{\partial \bar{\eta}_i \partial \eta_i} (\det \mathbf{A}) e^{\sum_{ij} \bar{\eta}_i A_{ij}^{-1} \eta_j} \\
&= (\det \mathbf{A}) (\mathbf{A}^{-1})
\end{aligned}$$

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B. FERMION COHERENT STATES

- (1) We require that Grassmann numbers anticommute with the Fermion operators,

$$\{\psi_i, \hat{c}_j\} = \{\psi_i, \hat{c}_j^\dagger\} = \{\bar{\psi}_i, \hat{c}_j\} = \{\bar{\psi}_i, \hat{c}_j^\dagger\} = 0$$

Define the Fermion coherent states:

$$|\psi\rangle \equiv |\psi_1, \dots, \psi_N\rangle = e^{-\sum_{i=1}^N \psi_i \hat{c}_i^\dagger} |0\rangle = \prod_{i=1}^N e^{-\psi_i \hat{c}_i^\dagger} |0\rangle = \prod_{i=1}^N (1 - \psi_i \hat{c}_i^\dagger) |0\rangle$$

$$\langle\psi| \equiv \langle\psi_N, \dots, \psi_1| = \langle 0| e^{-\sum_{i=1}^N \hat{c}_i \bar{\psi}_i} = \langle 0| \prod_{i=1}^N e^{-\hat{c}_i \bar{\psi}_i} = \langle 0| \prod_{i=1}^N (1 - \hat{c}_i \bar{\psi}_i)$$

(2) Coherent states are eigenstates of the annihilation operator $\hat{c}_i$

PROOF:

$$\begin{aligned} \hat{c}_i |\psi\rangle &= \hat{c}_i e^{-\sum_{j=1}^N \psi_j \hat{c}_j^\dagger} |0\rangle \\ &= \hat{c}_i \prod_j (1 - \psi_j \hat{c}_j^\dagger) |0\rangle \\ &= \prod_{j \neq i} (1 - \psi_j \hat{c}_j^\dagger) \hat{c}_i (1 - \psi_i \hat{c}_i^\dagger) |0\rangle \end{aligned}$$

For the single state i :

$$\begin{aligned} \hat{c}_i (1 - \psi_i \hat{c}_i^\dagger) |0\rangle &= -\hat{c}_i \psi_i \hat{c}_i^\dagger |0\rangle \\ &= \psi_i \hat{c}_i \hat{c}_i^\dagger |0\rangle \\ &= \psi_i (\hat{c}_i^\dagger \hat{c}_i + 1) |0\rangle \\ &= \psi_i |0\rangle \end{aligned}$$

Since $\psi_i^2 = 0 \Rightarrow \psi_i |0\rangle = \psi_i |0\rangle - \psi_i^2 \hat{c}_i^\dagger |0\rangle$

$$\hat{c}_i (1 - \psi_i \hat{c}_i^\dagger) |0\rangle = \psi_i |0\rangle - \psi_i^2 \hat{c}_i^\dagger |0\rangle = \psi_i (1 - \psi_i \hat{c}_i^\dagger) |0\rangle$$

Thus, we get

$$\begin{aligned} \hat{c}_i |\psi\rangle &= \prod_{j \neq i} (1 - \psi_j \hat{c}_j^\dagger) \psi_i (1 - \psi_i \hat{c}_i^\dagger) |0\rangle \\ &= \psi_i \prod_j (1 - \psi_j \hat{c}_j^\dagger) |0\rangle \\ &= \psi_i e^{-\sum_{j=1}^N \psi_j \hat{c}_j^\dagger} |0\rangle \\ &= \psi_i |\psi\rangle \end{aligned}$$

Similarly,

$$\langle\psi| \hat{c}_i^\dagger = \langle\psi| \bar{\psi}_i$$

Action of $\hat{c}_i^\dagger$ on a coherent state is a derivative

$$\begin{aligned}
 \hat{c}_i^\dagger |\psi\rangle &= \hat{c}_i^\dagger \left(1 - \psi_i \hat{c}_i^\dagger\right) \prod_{j \neq i} \left(1 - \psi_j \hat{c}_j^\dagger\right) |0\rangle \\
 &= \hat{c}_i^\dagger \prod_{j \neq i} \left(1 - \psi_j \hat{c}_j^\dagger\right) |0\rangle \\
 &= -\frac{\partial}{\partial \psi_i} \left(1 - \psi_i \hat{c}_i^\dagger\right) \prod_{j \neq i} \left(1 - \psi_j \hat{c}_j^\dagger\right) |0\rangle \\
 &= -\frac{\partial}{\partial \psi_i} |\psi\rangle
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \hat{c}_i |\psi\rangle &= \psi_i |\psi\rangle, & \hat{c}_i^\dagger |\psi\rangle &= -\frac{\partial}{\partial \psi_i} |\psi\rangle \\
 \langle \psi | \hat{c}_i^\dagger &= \langle \psi | \bar{\psi}_i, & \langle \psi | \hat{c}_i &= \frac{\partial}{\partial \psi_i} \langle \psi |
 \end{aligned}$$

(3) The inner product of two Fermion coherent states

$$\begin{aligned}
 \langle \psi' | \psi \rangle &= \left(\langle 0 | \prod_{i=1}^N (1 - \hat{c}_i \bar{\psi}'_i) \right) \left(\prod_{j=1}^N (1 - \psi_j \hat{c}_j^\dagger) | 0 \rangle \right) \\
 &= \prod_{i,j}^N \langle 0 | 1 - \hat{c}_i \bar{\psi}'_i - \psi_j \hat{c}_j^\dagger + \hat{c}_i \bar{\psi}'_i \psi_j \hat{c}_j^\dagger | 0 \rangle \\
 &= \prod_{i,j}^N \langle 0 | 1 + \bar{\psi}'_i \psi_j \hat{c}_i \hat{c}_j^\dagger | 0 \rangle \\
 &= \prod_{i,j}^N \langle 0 | 1 + \bar{\psi}'_i \psi_j (\hat{c}_j^\dagger \hat{c}_i + \delta_{ij}) | 0 \rangle \\
 &= \prod_{i=1}^N \langle 0 | 1 + \bar{\psi}'_i \psi_i | 0 \rangle \\
 &= \prod_{i=1}^N \langle 0 | e^{\bar{\psi}'_i \psi_i} | 0 \rangle \\
 &= e^{\sum_i \bar{\psi}'_i \psi_i}
 \end{aligned}$$

⇒ The set of Fermion coherent states is non-orthogonal.

OS:

As a consequence of their nonorthogonality, any Fermion coherent state can be expanded in terms of all the other Fermion coherent states. So the Fermion coherent states are not linearly independent.

(4) The resolution of identity for the Fermion coherent states reads,

$$\int \prod_i d\bar{\psi}_i d\psi_i e^{-\sum_i \bar{\psi}_i \psi_i} |\psi\rangle\langle\psi| = \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle\langle\psi| = \hat{1}$$

Thus, the set of Fermion coherent states is "overcomplete".

PROOF:

Using $|\psi\rangle = |0\rangle - \psi|1\rangle$, $\langle\psi| = \langle 0| - \psi\langle 1|$, we have

$$\begin{aligned} \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle\langle\psi| &= \int d\bar{\psi} d\psi (1 - \bar{\psi}\psi)(|0\rangle - \psi|1\rangle)(\langle 0| - \bar{\psi}\langle 1|) \\ &= \int d\bar{\psi} d\psi (1 - \bar{\psi}\psi) \\ &\quad \times (|0\rangle\langle 0| - \psi|1\rangle\langle 0| - \bar{\psi}|0\rangle\langle 1| + \psi\bar{\psi}|1\rangle\langle 1|) \end{aligned}$$

Since

$$\int d\bar{\psi} d\psi = 0, \quad \int d\bar{\psi} d\psi \psi = \int d\bar{\psi} d\psi \bar{\psi} = 0$$

we obtain

$$\begin{aligned} \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle\langle\psi| &= \int d\bar{\psi} d\psi (1 - \bar{\psi}\psi)(|0\rangle\langle 0| + \psi\bar{\psi}|1\rangle\langle 1|) \\ &= \int d\bar{\psi} d\psi (-\bar{\psi}\psi|0\rangle\langle 0| + \psi\bar{\psi}|1\rangle\langle 1|) \end{aligned}$$

Since

$$\int d\bar{\psi} d\psi \bar{\psi}\psi = - \int d\bar{\psi} d\psi \psi\bar{\psi} = -1$$

we obtain

$$\int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle\langle\psi| = |0\rangle\langle 0| + |1\rangle\langle 1| = \hat{1}$$

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3-2 Field Integral

A. FUNCTIONAL INTEGRAL REPRESENTATION OF PARTITION FUNCTION

(1) The partition function

$$Z = \text{Tr} e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_n \langle n | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle$$

where $|n\rangle = |n_1, n_2, \dots, n_N\rangle = \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger |0\rangle$ are bases in the Fock space. Insert the resolution of identity in terms of Fermion coherent states,

$$\begin{aligned} Z &= \sum_n \langle n | \underbrace{\int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle \langle \psi|}_{\hat{1}} e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \\ &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \sum_n \langle n | \psi \rangle \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \end{aligned}$$

where $|\psi\rangle \equiv |\psi_1, \dots, \psi_N\rangle$

Here,

$$\langle n | \psi \rangle = \langle 0 | \hat{c}_N \dots \hat{c}_2 \hat{c}_1 | \psi \rangle = \psi_N \dots \psi_2 \psi_1 \langle 0 | \psi \rangle$$

$$\text{Let } e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle = \sum_\ell a_\ell | \ell \rangle = \sum_\ell a_\ell \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger | 0 \rangle_\ell$$

Thus, we obtain

$$\begin{aligned} \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle &= \langle \psi | \sum_\ell a_\ell | \ell \rangle \\ &= \sum_\ell a_\ell \langle \psi | \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger | 0 \rangle_\ell \\ &= \sum_\ell a_\ell \bar{\psi}_1 \bar{\psi}_2 \dots \bar{\psi}_N \langle \psi | 0 \rangle_\ell \end{aligned}$$

Since

$$\begin{aligned} \langle \psi | 0 \rangle_\ell &= \left(\langle 0 | \prod_{i=1}^N (1 - \hat{c}_i \bar{\psi}_i) \right) | 0 \rangle_\ell = \langle 0 | 0 \rangle_\ell = 1 \\ \langle 0 | \psi \rangle &= \langle 0 | \left(\prod_{i=1}^N (1 - \psi_i \hat{c}_i^\dagger) \right) | 0 \rangle = \langle 0 | 0 \rangle = 1 \end{aligned}$$

we obtain

$$\langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle = \sum_{\ell} a_{\ell} \langle \psi | \hat{c}_1^{\dagger} \hat{c}_2^{\dagger} \cdots \hat{c}_N^{\dagger} | 0 \rangle = \sum_{\ell} a_{\ell} \bar{\psi}_1 \bar{\psi}_2 \cdots \bar{\psi}_N$$

$$\langle n | \psi \rangle = \langle 0 | \hat{c}_N \cdots \hat{c}_2 \hat{c}_1 | \psi \rangle = \psi_N \cdots \psi_2 \psi_1$$

Thus, we obtain

$$\begin{aligned} \langle n | \psi \rangle \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle &= \sum_{\ell} a_{\ell} \psi_N \cdots \psi_2 \psi_1 \bar{\psi}_1 \bar{\psi}_2 \cdots \bar{\psi}_N \\ &= \sum_{\ell} a_{\ell} \psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2 \cdots \psi_N \bar{\psi}_N \\ &= \sum_{\ell} a_{\ell} (-\bar{\psi}_1 \psi_1) (-\bar{\psi}_2 \psi_2) \cdots (-\bar{\psi}_N \psi_N) \\ &= \sum_{\ell} a_{\ell} (-\bar{\psi}_1) (-\bar{\psi}_2) \cdots \psi_2 \psi_1 \\ &= \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \langle n | \psi \rangle \end{aligned}$$

The partition function in terms of coherent states:

$$\begin{aligned} Z &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \sum_n \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \langle n | \psi \rangle \\ &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} \underbrace{\sum_n | n \rangle \langle n |}_{=\hat{1}} | \psi \rangle \\ &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi \rangle \end{aligned}$$

(2) Dividing the "imaginary-time interval" β into M small steps

$$\begin{aligned} e^{-\beta(\hat{H}-\mu\hat{N})} &= \left(e^{-\epsilon(\hat{H}-\mu\hat{N})} \right)^M, \text{ where } \epsilon = \frac{\beta}{M} \\ Z &= \int d\bar{\psi}^0 d\psi^0 e^{-\bar{\psi}^0 \psi^0} \langle -\psi^0 | \underbrace{e^{-\epsilon(\hat{H}-\mu\hat{N})} \cdots e^{-\epsilon(\hat{H}-\mu\hat{N})}}_{M \text{ times}} | \psi^0 \rangle \end{aligned}$$

Insert $M - 1$ resolutions of identity

$$\begin{aligned}
Z &= \int d\bar{\psi}^0 d\psi^0 e^{-\bar{\psi}^0 \psi^0} \langle -\psi^0 | \\
&\quad \times e^{-\epsilon(\hat{H}-\mu\hat{N})} \underbrace{\int d\bar{\psi}^{M-1} d\psi^{M-1} e^{-\bar{\psi}^{M-1} \psi^{M-1}} |\psi^{M-1}\rangle \langle \psi^{M-1}|}_{\hat{1}} \\
&\quad \times \dots \\
&\quad \times e^{-\epsilon(\hat{H}-\mu\hat{N})} \underbrace{\int d\bar{\psi}^1 d\psi^1 e^{-\bar{\psi}^1 \psi^1} |\psi^1\rangle \langle \psi^1|}_{\hat{1}} e^{-\epsilon(\hat{H}-\mu\hat{N})} \\
&\quad \times |\psi^0\rangle \\
&= \int d\bar{\psi}^0 d\psi^0 d\bar{\psi}^{M-1} d\psi^{M-1} \dots d\bar{\psi}^1 d\psi^1 e^{-\bar{\psi}^0 \psi^0} e^{-\bar{\psi}^{M-1} \psi^{M-1}} \dots e^{-\bar{\psi}^1 \psi^1} \\
&\quad \times \langle -\psi^0 | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^{M-1}\rangle \langle \psi^{M-1} | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^{M-2}\rangle \dots \\
&\quad \times \langle \psi^1 | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^0\rangle
\end{aligned}$$

Using the antiperiodic conditions:

$$\left. \begin{aligned} \bar{\psi}^M &= -\bar{\psi}^0 \\ \psi^M &= -\psi^0 \end{aligned} \right\} \Rightarrow \langle -\psi^0 | = \langle \psi^M |$$

we obtain

$$\begin{aligned}
Z &= \int d\bar{\psi}^M d\psi^M d\bar{\psi}^{M-1} d\psi^{M-1} \dots d\bar{\psi}^1 d\psi^1 e^{-\bar{\psi}^M \psi^M} e^{-\bar{\psi}^{M-1} \psi^{M-1}} \dots e^{-\bar{\psi}^1 \psi^1} \\
&\quad \times \langle \psi^M | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^{M-1}\rangle \langle \psi^{M-1} | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^{M-2}\rangle \dots \\
&\quad \times \langle \psi^1 | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^0\rangle \\
&= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-\sum_{m=1}^M \bar{\psi}^m \psi^m} \prod_{m=1}^M \langle \psi^m | e^{-\epsilon(\hat{H}-\mu\hat{N})} |\psi^{m-1}\rangle
\end{aligned}$$

In the limit of infinitesimal ϵ time interval

$$e^{-\epsilon(\hat{H}-\mu\hat{N})} \approx 1 - \epsilon(\hat{H} - \mu\hat{N}) = 1 - \epsilon(\hat{H}[\hat{c}^\dagger, \hat{c}] - \mu\hat{N}[\hat{c}^\dagger, \hat{c}])$$

where $\hat{H}[\hat{c}^\dagger, \hat{c}]$ and $\hat{N}[\hat{c}^\dagger, \hat{c}]$ are functions of operators $\hat{c}^\dagger$ and $\hat{c}$.

Thus, we obtain

$$\begin{aligned}
&\langle \psi^m | 1 - \epsilon(\hat{H} - \mu\hat{N}) | \psi^{m-1} \rangle \\
&= \langle \psi^m | 1 - \epsilon(\hat{H}[\hat{c}^\dagger, \hat{c}] - \mu\hat{N}[\hat{c}^\dagger, \hat{c}]) | \psi^{m-1} \rangle \\
&= \langle \psi^m | 1 - \epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}]) | \psi^{m-1} \rangle \\
&= \langle \psi^m | \psi^{m-1} \rangle (1 - \epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}])) \\
&= e^{\bar{\psi}^m \psi^{m-1}} e^{-\epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}])}
\end{aligned}$$

where $H[\bar{\psi}^m, \psi^{m-1}]$ and $N[\bar{\psi}^m, \psi^{m-1}]$ are functional of $\bar{\psi}^m$ and ψ^{m-1} .

Putting together all factors, we end up with

$$\begin{aligned}
Z &= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-\sum_{m=1}^M \bar{\psi}^m \psi^m} \\
&\quad \times \prod_{m=1}^M e^{\bar{\psi}^m \psi^{m-1}} e^{-\epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}])} \\
&= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{\sum_{m=1}^M -\bar{\psi}^m \psi^m + \bar{\psi}^m \psi^{m-1} - \epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}])} \\
&= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-\sum_{m=1}^M \bar{\psi}^m (\psi^m - \psi^{m-1}) + \epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}])} \\
&= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-S[\bar{\psi}^m, \psi^m]}
\end{aligned}$$

where

$$S[\bar{\psi}^m, \psi^m] = \sum_{m=1}^M \bar{\psi}^m (\psi^m - \psi^{m-1}) + \epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}])$$

(3) Continuum limit $M \rightarrow \infty$. We can then (symbolically) write

$$\begin{aligned}
S[\bar{\psi}^m, \psi^m] &= \sum_{m=1}^M \bar{\psi}^m \epsilon \frac{(\psi^m - \psi^{m-1})}{\epsilon} + \epsilon(H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}]) \\
&= \epsilon \sum_{m=1}^M \bar{\psi}^m \frac{(\psi^m - \psi^{m-1})}{\epsilon} + H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}]
\end{aligned}$$

$$\text{Let } \epsilon \sum_{m=1}^M \rightarrow \int_0^\beta d\tau, \text{ and } \bar{\psi}^m \frac{(\psi^m - \psi^{m-1})}{\epsilon} \rightarrow \bar{\psi}(\tau) \frac{\partial}{\partial \tau} \psi(\tau)$$

NOTE:

The notion of replacing differences by derivatives is purely symbolic for Grassmann variables. There is no numerical values for $\psi^m - \psi^{m-1}$.

We obtain

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \left\{ \bar{\psi}(\tau) \frac{\partial}{\partial \tau} \psi(\tau) + H[\bar{\psi}(\tau), \psi(\tau)] - \mu N[\bar{\psi}(\tau), \psi(\tau)] \right\}$$

with the antiperiodic conditions $\bar{\psi}(\beta) = -\bar{\psi}(0)$ and $\psi(\beta) = -\psi(0)$,

inherited from $\bar{\psi}^M = -\bar{\psi}^0$ and $\psi^M = -\psi^0$.

$$Z = \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-S[\bar{\psi}, \psi]} = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]}$$

- (4) Functional integral for the partition function of a many-particle system,

$$\begin{aligned} Z &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \left\langle -\psi \left| e^{-\beta(\hat{H} - \mu\hat{N})} \right| \psi \right\rangle \dots (a) \\ &= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-S[\bar{\psi}^m, \psi^m]} \dots (b) \\ &= \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]} \dots (c) \end{aligned}$$

where

$$\begin{aligned} S[\bar{\psi}^m, \psi^m] &= \sum_{m=1}^M \bar{\psi}^m (\psi^m - \psi^{m-1}) - \epsilon (H[\bar{\psi}^m, \psi^{m-1}] - \mu N[\bar{\psi}^m, \psi^{m-1}]) \\ S[\bar{\psi}, \psi] &= \int_0^\beta d\tau \left\{ \bar{\psi}(\tau) \frac{\partial}{\partial \tau} \psi(\tau) + H[\bar{\psi}(\tau), \psi(\tau)] - \mu N[\bar{\psi}(\tau), \psi(\tau)] \right\} \end{aligned}$$

- (5) The action of fermion fields

$$\begin{aligned} \hat{H} &= \sum_k \varepsilon_k \hat{c}_k^\dagger(\tau) \hat{c}_k(\tau) + \frac{1}{2} \sum_{kk'q} V(q) \hat{c}_{k+q}^\dagger(\tau) \hat{c}_{k'-q}^\dagger(\tau) \hat{c}_{k'}(\tau) \hat{c}_k(\tau) \\ \hat{N} &= \sum_k \hat{c}_k^\dagger(\tau) \hat{c}_k(\tau) \end{aligned}$$

Using that

$$\begin{aligned} \hat{c}_k |\psi\rangle &= \psi_k |\psi\rangle \\ \langle \psi | \hat{c}_k^\dagger &= \langle \psi | \bar{\psi}_k \end{aligned}$$

They are written in terms of Grassmann variables.

$$\begin{aligned} H &= \sum_k \varepsilon_k \bar{\psi}_k(\tau) \psi_k(\tau) + \frac{1}{2} \sum_{kk'q} V(q) \bar{\psi}_{k+q}(\tau) \bar{\psi}_{k'-q}(\tau) \psi_{k'}(\tau) \psi_k(\tau) \\ N &= \sum_k \bar{\psi}_k(\tau) \psi_k(\tau) \end{aligned}$$

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \left\{ \sum_k \bar{\psi}_k(\tau) \frac{\partial}{\partial \tau} \psi_k(\tau) + \varepsilon_k \bar{\psi}_k(\tau) \psi_k(\tau) - \mu \bar{\psi}_k(\tau) \psi_k(\tau) \right. \\ \left. + \frac{1}{2} \sum_{kk'q} V(q) \bar{\psi}_{k+q}(\tau) \bar{\psi}_{k'-q}(\tau) \psi_{k'}(\tau) \psi_k(\tau) \right\}$$

Let $\xi_k = \varepsilon_k - \mu$

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \left\{ \sum_k \bar{\psi}_k(\tau) \left[\frac{\partial}{\partial \tau} + \xi_k \right] \psi_k(\tau) \right. \\ \left. + \frac{1}{2} \sum_{kk'q} V(q) \bar{\psi}_{k+q}(\tau) \bar{\psi}_{k'-q}(\tau) \psi_{k'}(\tau) \psi_k(\tau) \right\}$$

B. FREE FERMION FIELDS

(1) The action of a free-fermion system,

$$H = \sum_k \varepsilon_k \bar{\psi}_k(\tau) \psi_k(\tau) \\ S[\bar{\psi}, \psi] = \sum_k \int_0^\beta d\tau \left\{ \bar{\psi}_k(\tau) \left[\frac{\partial}{\partial \tau} + \xi_k \right] \psi_k(\tau) \right\}$$

Fourier series expansion of $\psi_k(\tau)$ and the inversion of $\psi_k(i\omega_n)$ are

$$\psi_k(\tau) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} e^{-i\omega_n \tau} \psi_k(i\omega_n) \\ \psi_k(i\omega_n) = \frac{1}{\sqrt{\beta}} \int_0^\beta e^{i\omega_n \tau} \psi_k(\tau) d\tau$$

where $\omega_n = \frac{(2n+1)\pi}{\beta\hbar}$

$$S[\bar{\psi}, \psi] = \sum_{k,n,n'} \frac{1}{\beta} \int_0^\beta e^{-i(\omega_n - \omega'_n)\tau} d\tau \left\{ \bar{\psi}_k(i\omega_n) (-i\omega'_n + \xi_k) \psi_k(i\omega'_n) \right\} \\ = \sum_{k,n,n'} \frac{1}{\beta} \beta \delta(\omega_n - \omega'_n) \left\{ \bar{\psi}_k(i\omega_n) (-i\omega'_n + \xi_k) \psi_k(i\omega'_n) \right\} \\ = \sum_{k,n} \bar{\psi}_k(i\omega_n) (-i\omega_n + \xi_k) \psi_k(i\omega_n)$$

(2) Since the transformation $\psi(\tau) \rightarrow \psi(i\omega_n)$ has unit Jacobian, one obtains

$$\begin{aligned}
Z &= \int D[\bar{\psi}, \psi] e^{-\sum_{k,n} \bar{\psi}_k(i\omega_n)(-i\omega_n + \xi_k)\psi_k(i\omega_n)} \\
&= \prod_{k,n} \int D[\bar{\psi}, \psi] e^{-\bar{\psi}_k(i\omega_n)(-i\omega_n + \xi_k)\psi_k(i\omega_n)} \\
&= \prod_{k,n} (-i\omega_n + \xi_k)
\end{aligned}$$

(3) The free energy is

$$F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \sum_{k,n} \ln(-i\omega_n + \xi_k)$$

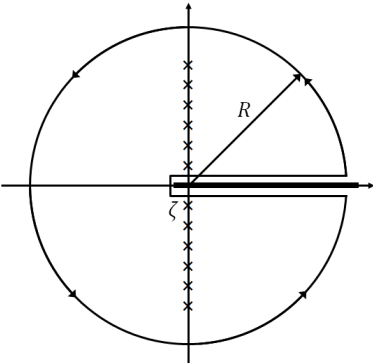
First calculate the summation of Matsubara frequencies,

$$\sum_n \ln(-i\omega_n + \xi_k) e^{i\omega_n \eta}$$

We now perform the sum by the method of contour integration

$$I = \frac{1}{2\pi i} \oint_C f(z) \ln(-z + \zeta) e^{z\eta} dz$$

where the contour C is chosen such as to avoid the branch cut of the function $\ln(-z + \zeta)$.



Evaluate the integral along the contour C directly:

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} f(\epsilon) \ln(-\epsilon + i\eta + \zeta) d\epsilon + \frac{1}{2\pi i} \int_{\infty}^{\zeta^-} f(\epsilon) \ln(-\epsilon - i\eta + \zeta) d\epsilon \\
&= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} f(\epsilon) \left[\ln(-\epsilon + i\eta + \zeta) - \ln(-\epsilon - i\eta + \zeta) \right] d\epsilon
\end{aligned}$$

where $\zeta^- = \zeta - 0^+$

Use

$$f(\epsilon) = \frac{1}{e^\epsilon + 1} = \frac{e^{-\epsilon}}{1 + e^{-\epsilon}} = -\frac{d}{d\epsilon} \ln|1 + e^{-\epsilon}|$$

Integral by parts

$$\begin{aligned} I &= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} \left(-\frac{d}{d\epsilon} \ln|1 + e^{-\epsilon}| \right) \left[\ln(-\epsilon + i\eta + \zeta) - \ln(-\epsilon - i\eta + \zeta) \right] d\epsilon \\ &= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} \ln|1 + e^{-\epsilon}| \frac{d}{d\epsilon} \left[\ln(-\epsilon + i\eta + \zeta) - \ln(-\epsilon - i\eta + \zeta) \right] d\epsilon \\ &= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} \ln|1 + e^{-\epsilon}| \left[\frac{-1}{-\epsilon + i\eta + \zeta} - \frac{-1}{-\epsilon - i\eta + \zeta} \right] d\epsilon \\ &= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} \ln|1 + e^{-\epsilon}| \left[\frac{1}{\epsilon - \zeta - i\eta} - \frac{1}{\epsilon - \zeta + i\eta} \right] d\epsilon \end{aligned}$$

Since $\frac{1}{\epsilon - \zeta \mp i\eta} = \mathcal{P} \frac{1}{\epsilon - \zeta} \pm i\pi\delta(\epsilon - \zeta)$

$$\begin{aligned} I &= \frac{1}{2\pi i} \int_{\zeta^-}^{\infty} \ln|1 + e^{-\epsilon}| \left[i\pi\delta(\epsilon - \zeta) + i\pi\delta(\epsilon - \zeta) \right] d\epsilon \\ &= \int_{\zeta^-}^{\infty} \ln|1 + e^{-\epsilon}| \delta(\epsilon - \zeta) d\epsilon \\ &= \ln(1 + e^{-\zeta}) \end{aligned}$$

Thus, we have

$$\begin{aligned} \sum_n \ln(-i\omega_n + \xi_k) e^{i\omega_n \eta} \Big|_{\eta=0} &= \ln(1 + e^{-\beta\xi_k}) \\ F &= -\frac{1}{\beta} \sum_{k,n} \ln(-i\omega_n + \xi_k) = -\frac{1}{\beta} \sum_k \ln(1 + e^{-\beta\xi_k}) \end{aligned}$$

C. APPENDIX: SOLUTIONS OF PARTITION FUNCTION

I. From equation (a):

$$\begin{aligned} Z &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \left\langle -\psi \left| e^{-\beta(\hat{H} - \mu\hat{N})} \right| \psi \right\rangle \\ &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta \sum_k \xi_k \hat{c}_k^\dagger \hat{c}_k} | \psi \rangle \\ &= \prod_k \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta \xi_k \hat{c}_k^\dagger \hat{c}_k} | \psi \rangle \end{aligned}$$

We cannot simply replace $\hat{c}_k^\dagger$ and $\hat{c}_k$ by $-\bar{\psi}_k$ and ψ_k respectively in the exponential. This is because when we expand out the exponential, not

all the $\hat{c}_k$ will be acting to the right on $|\psi\rangle$ and neither will all $\hat{c}_k^\dagger$ be acting to the left on $\langle -\psi|$. We need to convert the exponential to its *normal-ordered* form in which all the creation operators stand to the left and all the annihilation operators to the right.

$$e^{-\beta \xi_k \hat{c}_k^\dagger \hat{c}_k} = 1 - \beta \xi_k \hat{c}_k^\dagger \hat{c}_k + \frac{1}{2!} \beta^2 \xi_k^2 (\hat{c}_k^\dagger \hat{c}_k)^2 - \dots$$

Since

$$\begin{aligned} (\hat{c}_k^\dagger \hat{c}_k)^2 &= \hat{c}_k^\dagger \hat{c}_k \hat{c}_k^\dagger \hat{c}_k = \hat{c}_k^\dagger (1 - \hat{c}_k^\dagger \hat{c}_k) \hat{c}_k = \hat{c}_k^\dagger \hat{c}_k \\ &\vdots \\ e^{-\beta \xi_k \hat{c}_k^\dagger \hat{c}_k} &= 1 - \beta \xi_k \hat{c}_k^\dagger \hat{c}_k + \frac{1}{2!} \beta^2 \xi_k^2 \hat{c}_k^\dagger \hat{c}_k - \dots \\ &= 1 + \left(-\beta \xi_k + \frac{1}{2!} \beta^2 \xi_k^2 - \dots \right) \hat{c}_k^\dagger \hat{c}_k \\ &= 1 + (e^{-\beta \xi_k} - 1) \hat{c}_k^\dagger \hat{c}_k \end{aligned}$$

we obtain

$$Z = \prod_k \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\psi | \psi \rangle (1 + (e^{-\beta \xi_k} - 1)(-\bar{\psi}\psi))$$

Since

$$\begin{aligned} \langle -\psi | \psi \rangle &= \langle 0 | e^{\sum_k \hat{c}_k \bar{\psi}_k} e^{-\sum_k \psi_k \hat{c}_k^\dagger} | 0 \rangle \\ &= \langle 0 | \prod_k (1 + \hat{c}_k \bar{\psi}_k) (1 - \psi_k \hat{c}_k^\dagger) | 0 \rangle \\ &= \langle 0 | \prod_k (1 - \bar{\psi}_k \psi_k \hat{c}_k \hat{c}_k^\dagger) | 0 \rangle \\ &= \prod_k e^{-\bar{\psi}_k \psi_k} \\ &= e^{-\sum_k \bar{\psi}_k \psi_k} \\ &= e^{-\bar{\psi}\psi} \end{aligned}$$

we obtain

$$\begin{aligned}
Z &= \prod_k \int d\bar{\psi} d\psi e^{-2\bar{\psi}\psi} \left(1 + (e^{-\beta\xi_k} - 1)(-\bar{\psi}\psi)\right) \\
&= \prod_k \int d\bar{\psi} d\psi (1 - 2\bar{\psi}\psi) \left(1 + (e^{-\beta\xi_k} - 1)(-\bar{\psi}\psi)\right) \\
&= \prod_k \int d\bar{\psi} d\psi \left(-(1 + e^{-\beta\xi_k})\bar{\psi}\psi\right) \\
&= \prod_k (1 + e^{-\beta\xi_k})
\end{aligned}$$

Take logarithm of Z to obtain the free energy

$$F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \sum_k \ln(1 + e^{-\beta\xi_k})$$

II. From equation (b):

$$\begin{aligned}
Z &= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-S[\bar{\psi}^m, \psi^m]} \\
S[\bar{\psi}^m, \psi^m] &= \sum_{m=1}^M [\bar{\psi}^m(\psi^m - \psi^{m-1}) + \epsilon\xi_k \bar{\psi}^m \psi^{m-1}] \\
&= \sum_{m=1}^M [\bar{\psi}^m \psi^m + (\epsilon\xi_k - 1)\bar{\psi}^m \psi^{m-1}]
\end{aligned}$$

Fourier expansion

$$\begin{aligned}
\psi^m &= \frac{1}{\sqrt{M}} \sum_{n=1}^M a_n e^{-i\omega_n m \epsilon}, \quad \bar{\psi}^m = \frac{1}{\sqrt{M}} \sum_{n=1}^M \bar{a}_n e^{i\omega_n m \epsilon} \\
S &= \frac{1}{M} \sum_{m=1}^M \sum_{n=1}^M [\bar{a}_n a_n + (\epsilon\xi_k - 1)\bar{a}_n a_n e^{i\epsilon\omega_n}] = \sum_{n=1}^M \bar{a}_n a_n [1 - e^{-\epsilon\xi_k} e^{i\epsilon\omega_n}]
\end{aligned}$$

Since the transformation $\psi \rightarrow a$ has unit Jacobian, one obtains

$$\begin{aligned}
Z &= \int \prod_k \prod_{n=1}^M d\bar{a}_{k,n} da_{k,n} e^{-\sum_n \bar{a}_{k,n} a_{k,n} [1 - e^{-\epsilon\xi_k} e^{i\epsilon\omega_n}]} \\
&= \prod_k \prod_{n=1}^M [1 - e^{-\epsilon\xi_k} e^{i\epsilon\omega_n}]
\end{aligned}$$

Since

$$\prod_{n=1}^M (1 - e^{-\epsilon \xi_k} e^{i\epsilon \omega_n}) \approx 1 + e^{-M\epsilon \xi_k} = 1 + e^{-\beta \xi_k}$$

$$\text{with } \omega_n = \frac{(2n+1)\pi}{\beta} \Rightarrow e^{i\epsilon \omega_n} = e^{\frac{i(2n+1)\pi}{M}} = -1$$

we obtain

$$Z = \prod_k (1 + e^{-\beta \xi_k})$$

- Alternative approach:

Let $a = 1 - \epsilon \xi_k$

$$\boldsymbol{\Psi}_k = \begin{pmatrix} \psi_k^1 \\ \psi_k^2 \\ \vdots \\ \psi_k^M \end{pmatrix}, \quad \mathbf{S}_k = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -a & 1 & 0 & \cdots \\ 0 & -a & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$Z = \int \prod_k d\bar{\boldsymbol{\Psi}}_k d\boldsymbol{\Psi}_k e^{-\bar{\boldsymbol{\Psi}}_k \cdot \mathbf{S}_k \cdot \boldsymbol{\Psi}_k}$$

Make a unitary similarity transformation to diagonalize $\mathbf{S}$

$$\mathbf{D}_k = \mathbf{U}^\dagger \mathbf{S}_k \mathbf{U}$$

Let $\bar{\boldsymbol{\eta}}_k = \bar{\boldsymbol{\Psi}}_k \mathbf{U}$, $\boldsymbol{\eta}_k = \mathbf{U}^\dagger \boldsymbol{\Psi}_k$

$$\bar{\boldsymbol{\eta}}_k \cdot \mathbf{D}_k \cdot \boldsymbol{\eta}_k = \bar{\boldsymbol{\eta}}_k \cdot \mathbf{U}^\dagger \mathbf{S}_k \mathbf{U} \cdot \boldsymbol{\eta}_k = \bar{\boldsymbol{\Psi}}_k \cdot \mathbf{S}_k \cdot \boldsymbol{\Psi}_k$$

The Jacobian is $|\det \mathbf{U}| = 1$

$$Z = \int \prod_k d\bar{\boldsymbol{\eta}}_k d\boldsymbol{\eta}_k e^{-\bar{\boldsymbol{\eta}}_k \cdot \mathbf{D}_k \cdot \boldsymbol{\eta}_k} = \prod_k \prod_{m=1}^M D_{mm} = \prod_k \det \mathbf{D}_k = \prod_k \det \mathbf{S}_k$$

$$\det \mathbf{S}_k = (1 - (-1)^{M-1} (-a)^M) = (1 + (1 - \epsilon \xi_k)^M)$$

As $M \rightarrow \infty$

$$\det \mathbf{S}_k = 1 + e^{-\beta \xi_k}$$

$$Z = \prod_k (1 + e^{-\beta \xi_k})$$

Take logarithm of Z to obtain the free energy

$$F = -\frac{1}{\beta} \sum_k \ln(1 + e^{-\beta \xi_k})$$

3-3 Generating Functionals

A. GENERATING FUNCTIONAL OF GREEN'S FUNCTIONS

(1) Matsubara Green's function

$$\begin{aligned}
 G(r, \tau; r', \tau') &= -\langle \hat{T} [\hat{\Psi}(r, \tau) \hat{\Psi}^\dagger(r', \tau')] \rangle \\
 &= -\sum_{\nu\nu'} \psi_{\nu'}^*(r') \psi_\nu(r) \left\langle \hat{T} [\hat{c}_\nu(\tau) \hat{c}_{\nu'}^\dagger(\tau')] \right\rangle \\
 &= \sum_{\nu\nu'} \psi_{\nu'}^*(r') \psi_\nu(r) G(\nu, \tau; \nu', \tau')
 \end{aligned}$$

where $G(\nu, \tau; \nu', \tau') = -\langle \hat{T} [\hat{c}_\nu(\tau) \hat{c}_{\nu'}^\dagger(\tau')] \rangle$.

In k-space, we have

$$\begin{aligned}
 G(r, \tau; r', \tau') &= \frac{1}{N} \sum_{kk'} e^{-ik'r'} e^{ikr} G(k, \tau; k', \tau') \\
 &= \frac{1}{N} \sum_{kk'} e^{-i(k'-k)r'} e^{ik(r-r')} G(k, \tau; k', \tau')
 \end{aligned}$$

Since $G(r, \tau; r', \tau') = G(r - r', \tau, \tau')$, the dependence on r' on the R.H.S. must vanish. Therefore, the k-space Green's function is nonzero only when $k = k'$, i.e., $G(k, \tau; k', \tau') = \delta_{k,k'} G(k, \tau, \tau')$. Thus we get

$$G(r - r', \tau, \tau') = \frac{1}{N} \sum_k e^{ik(r-r')} G(k, \tau, \tau')$$

where

$$\begin{aligned}
 G(k, \tau, \tau') &= -\langle \hat{T} [\hat{c}_k(\tau) \hat{c}_k^\dagger(\tau')] \rangle \\
 &= -\frac{1}{Z} \text{Tr} \left[e^{-\beta(\hat{H} - \mu \hat{N})} \hat{T} [\hat{c}_k(\tau) \hat{c}_k^\dagger(\tau')] \right] \\
 &= \begin{cases} -\frac{\text{Tr} [e^{-\beta(\hat{H} - \mu \hat{N})} \hat{c}_k(\tau) \hat{c}_k^\dagger(\tau')]}{\text{Tr} [e^{-\beta(\hat{H} - \mu \hat{N})}]}, & \text{for } \tau > \tau' \\ \frac{\text{Tr} [e^{-\beta(\hat{H} - \mu \hat{N})} \hat{c}_k^\dagger(\tau') \hat{c}_k(\tau)]}{\text{Tr} [e^{-\beta(\hat{H} - \mu \hat{N})}]}, & \text{for } \tau < \tau' \end{cases}
 \end{aligned}$$

(2) Write the Green's function as a functional integral

For $\tau > \tau'$

The numerator:

$$\begin{aligned}
& \text{Tr} \left[e^{-\beta(\hat{H}-\mu\hat{N})} \hat{c}_k(\tau) \hat{c}_k^\dagger(\tau') \right] \\
&= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \sum_k \langle k | \hat{c}_k(\tau) | \psi \rangle \langle \psi | \hat{c}_k^\dagger(\tau') e^{-\beta(\hat{H}-\mu\hat{N})} | k \rangle \\
&= \int d\bar{\psi} d\psi \psi(\tau) \bar{\psi}(\tau') e^{-\bar{\psi}\psi} \sum_k \langle k | \psi \rangle \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | k \rangle \\
&= \int d\bar{\psi} d\psi \psi(\tau) \bar{\psi}(\tau') e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} \underbrace{\sum_k | k \rangle \langle k |}_{\hat{1}} | \psi \rangle \\
&= \int d\bar{\psi} d\psi \psi(\tau) \bar{\psi}(\tau') e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi \rangle \\
&= \int D[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(\tau') e^{-S[\bar{\psi}, \psi]}
\end{aligned}$$

The denominator:

$$\begin{aligned}
& \text{Tr} \left[e^{-\beta(\hat{H}-\mu\hat{N})} \right] = \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \sum_k \langle k | \psi \rangle \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | k \rangle \\
&= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi \rangle \\
&= \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]} \\
& G(k, \tau, \tau') = - \frac{\int D[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(\tau') e^{-S[\bar{\psi}, \psi]}}{\int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]}}
\end{aligned}$$

For $\tau < \tau'$

$$\begin{aligned}
G(k, \tau, \tau') &= \frac{\int D[\bar{\psi}, \psi] \bar{\psi}(\tau') \psi(\tau) e^{-S[\bar{\psi}, \psi]}}{\int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]}} \\
&= - \frac{\int D[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(\tau') e^{-S[\bar{\psi}, \psi]}}{\int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]}}
\end{aligned}$$

Thus the Green's function can be written as a functional integral.

$$G(k, \tau, \tau') = - \frac{\int D[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(\tau') e^{-S[\bar{\psi}, \psi]}}{\int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]}} = - \langle \psi(\tau) \bar{\psi}(\tau') \rangle$$

(3) With the help of the identities

$$\bar{\psi}(\tau)e^{\int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))} = -\frac{\delta}{\delta\eta(\tau)}e^{\int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))}$$

$$\psi(\tau)e^{\int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))} = \frac{\delta}{\delta\bar{\eta}(\tau)}e^{\int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))}$$

where $\bar{\eta}$ and η are Grassmann sources that couple linearly to the fields $\bar{\psi}$ and ψ .

The two-point Green's function can then be generated as

$$G^{(2)}(\tau, \tau') = -\frac{1}{Z[0]} \int D[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(\tau') e^{-S[\bar{\psi}, \psi]}$$

$$= -\frac{1}{Z[0]} \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]} \frac{-\delta^2}{\delta\bar{\eta}(\tau)\delta\eta(\tau')} e^{\int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))} \Bigg|_{\substack{\bar{\eta}=0 \\ \eta=0}}$$

$$= \frac{1}{Z[0]} \frac{\delta^2}{\delta\bar{\eta}(\tau)\delta\eta(\tau')} \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi] + \int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))} \Bigg|_{\substack{\bar{\eta}=0 \\ \eta=0}}$$

where $Z[0] = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]}$

(4) We can define a generating functional of Green's functions as

$$Z[\bar{\eta}, \eta] = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi] + \int_0^\beta d\tau(\bar{\eta}(\tau)\psi(\tau)+\bar{\psi}(\tau)\eta(\tau))}$$

The Green's function can then be expressed in terms of functional derivatives,

$$G^{(2n)}(\tau_1, \dots, \tau'_1 \dots)$$

$$= \frac{1}{Z[\bar{\eta}, \eta]} \frac{\delta^{2n} Z[\bar{\eta}, \eta]}{\delta\bar{\eta}(\tau_1) \cdots \delta\bar{\eta}(\tau_n) \delta\eta(\tau'_n) \cdots \delta\eta(\tau'_1)} \Bigg|_{\bar{\eta}=\eta=0}$$

EXAMPLES:

1. The action functional of free fields is given by

$$S_0[\bar{\psi}, \psi] = \int_0^\beta d\tau \bar{\psi}(\tau) \left(\frac{\partial}{\partial \tau} + \xi_k \right) \psi(\tau)$$

$$\text{Let } G_0^{-1} = \frac{\partial}{\partial \tau} + \xi_k$$

$$S_0[\bar{\psi}, \psi] = \int_0^\beta d\tau \bar{\psi}(\tau) G_0^{-1} \psi(\tau)$$

2. The two-point Green's function in terms of functional integral:

$$\begin{aligned}
G(k, \tau) &= - \frac{\int D[\bar{\psi}, \psi] \psi(\tau) \bar{\psi}(\tau') e^{-\int_0^\beta d\tau \bar{\psi}(\tau) G_0^{-1} \psi(\tau)}}{\int D[\bar{\psi}, \psi] e^{-\int_0^\beta d\tau \bar{\psi}(\tau) G_0^{-1} \psi(\tau)}} \\
&= \frac{\det(G_0^{-1}) G_0}{\det(G_0^{-1})} \\
&= G_0
\end{aligned}$$

3. The two-point Green's function derived from the generating functional:

$$S_0[\bar{\psi}, \psi] = \int_0^\beta d\tau \bar{\psi}(\tau) G_0^{-1} \psi(\tau) = \int_0^\beta d\tau d\tau' \bar{\psi}(\tau) G_0^{-1} \delta(\tau - \tau') \psi(\tau')$$

The generating functional of free fields $Z_0[\bar{\eta}, \eta]$ is given by

$$\begin{aligned}
Z_0[\bar{\eta}, \eta] &= \int D[\bar{\psi}, \psi] \\
&\quad \times e^{-\int_0^\beta d\tau d\tau' \bar{\psi}(\tau) G_0^{-1} \delta(\tau - \tau') \psi(\tau') + \int_0^\beta d\tau (\bar{\eta}(\tau) \psi(\tau) + \bar{\psi}(\tau) \eta(\tau))} \\
&= \det(G_0^{-1}) e^{\int d\tau d\tau' \bar{\eta}(\tau) G_0 \delta(\tau - \tau') \eta(\tau')} \\
&= \det(G_0^{-1}) e^{\int d\tau \bar{\eta}(\tau) G_0 \eta(\tau)}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
G^{(2)}(\tau) &= \frac{1}{Z_0} \frac{\delta^2 Z[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau) \delta \eta(\tau)} \Bigg|_{\bar{\eta}=\eta=0} \\
&= \frac{\delta^2}{\delta \bar{\eta}(\tau) \delta \eta(\tau)} e^{\int d\tau \bar{\eta}(\tau) G_0 \eta(\tau)} \Bigg|_{\bar{\eta}=\eta=0} \\
&= G_0
\end{aligned}$$

B. GENERATING FUNCTIONAL OF CONNECTED GREEN'S FUNCTIONS

- (1) The link-cluster theorem:

We consider a many-body system with an interaction,

$$\begin{aligned}
S[\bar{\psi}, \psi] &= S_0[\bar{\psi}, \psi] + S_{\text{int}}[\bar{\psi}, \psi] \\
&= \int_0^\beta d\tau d\tau' \left\{ \bar{\psi}(\tau) \left(\frac{\partial}{\partial \tau} + \xi_k \right) \psi(\tau') - V[\bar{\psi}, \psi] \right\}
\end{aligned}$$

The interacting partition function and non-interacting partition function are

$$Z = \int D[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi] - S_{\text{int}}[\bar{\psi}, \psi]}$$

$$Z_0 = \int D[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi]}$$

Thus, we obtain

$$\begin{aligned} \frac{Z}{Z_0} &= \frac{\int D[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi]} e^{-S_{\text{int}}[\bar{\psi}, \psi]}}{\int D[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi]}} \\ &= \left\langle e^{-S_{\text{int}}[\bar{\psi}, \psi]} \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{\text{int}}^n[\bar{\psi}, \psi] \rangle \end{aligned}$$

The link-cluster theorem states that $\ln \frac{Z}{Z_0}$ is the sum of all topological nonequivalent **connected** diagrams.

$$\Rightarrow \ln \frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{\text{int}}^n[\bar{\psi}, \psi] \rangle_{\text{conn}} = \sum \text{connected diagrams}$$

- (2) We can define a generating functional of connected Green's functions as,

$$W[\bar{\eta}, \eta] = \ln Z[\bar{\eta}, \eta]$$

where the generating functional of Green's functions is

$$Z[\bar{\eta}, \eta] = \int D[\bar{\psi}, \psi] e^{-S_0[\bar{\psi}, \psi] - S_{\text{int}}[\bar{\psi}, \psi] + \int_0^\beta d\tau (\bar{\eta}(\tau) \psi(\tau) + \bar{\psi}(\tau) \eta(\tau))}$$

The connected Green's function can then be expressed in terms of functional derivatives,

$$G_c^{(2n)}(\tau_1, \dots, \tau'_1, \dots) = \frac{\delta^{2n} W[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau_1) \cdots \delta \bar{\eta}(\tau_n) \delta \eta(\tau'_1) \cdots \delta \eta(\tau'_n)} \Big|_{\bar{\eta}=\eta=0}$$

NOTE:

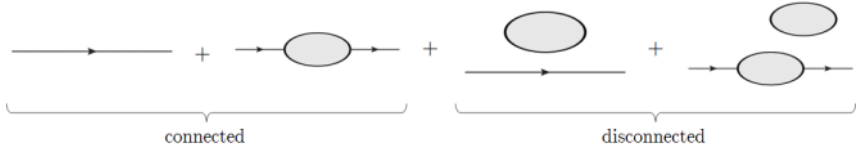
The generating functional $Z[\bar{\eta}, \eta]$ provides the full Green's function and the generating functional $W[\bar{\eta}, \eta]$ gives only the connected Green's functions.

EXAMPLES:

1. The two-point full Green's function

$$G^{(2)}(\tau, \tau') = \frac{1}{Z_0} \frac{\delta^2 Z[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau) \delta \eta(\tau')} \bigg|_{\bar{\eta}=\eta=0} = -\langle \psi(\tau) \bar{\psi}(\tau') \rangle$$

contains connected and disconnected diagrams, i.e.,



The two-point connected Green's function is given by

$$\begin{aligned} G_c^{(2)}(\tau, \tau') &= \frac{\delta^2 W[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau) \delta \eta(\tau')} \bigg|_{\bar{\eta}=\eta=0} \\ &= \frac{\delta}{\delta \bar{\eta}(\tau)} \left(\frac{1}{Z} \frac{\delta Z[\bar{\eta}, \eta]}{\delta \eta(\tau')} \right) \\ &= \frac{1}{Z} \frac{\delta^2 Z[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau) \delta \eta(\tau')} - \frac{1}{Z^2} \frac{\delta Z[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau)} \frac{\delta Z[\bar{\eta}, \eta]}{\delta \eta(\tau')} \\ &= -\langle \psi(\tau) \bar{\psi}(\tau') \rangle + \langle \psi(\tau) \rangle \langle \bar{\psi}(\tau') \rangle \end{aligned}$$

The second term precisely contains disconnected diagrams.

Therefore, in $G_c^{(2)}$ the disconnected pieces cancel and connected pieces remain.

- The generating functionals of free fields are

$$\begin{aligned} Z_0[\bar{\eta}, \eta] &= \det(G_0^{-1}) e^{\int d\tau \bar{\eta}(\tau) G_0 \eta(\tau)} \\ W_0[\bar{\eta}, \eta] &= \ln Z_0[\bar{\eta}, \eta] = \int d\tau d\tau' \bar{\eta}(\tau) G_0 \eta(\tau') \\ G_c^{(2)}(\tau, \tau') &= \frac{\delta^2 W[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau) \delta \eta(\tau')} \bigg|_{\bar{\eta}=\eta=0} = -G_0 \end{aligned}$$

The only connected Green's function, for free fields, is the two-point Green's function.

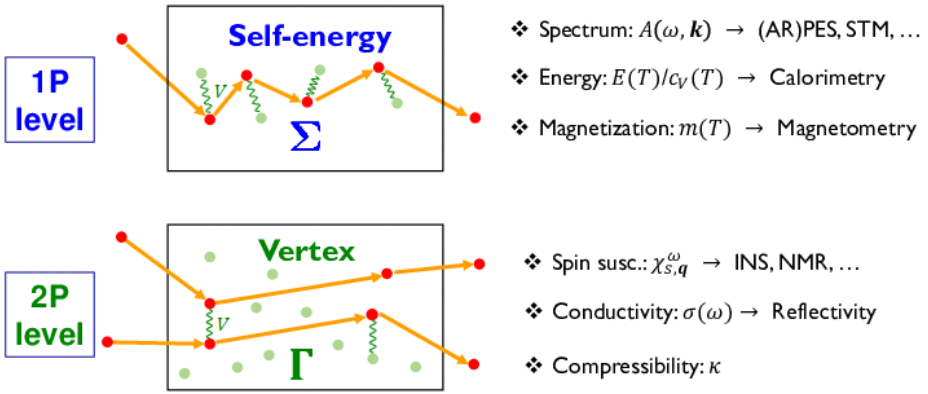
C. GENERATING FUNCTIONAL OF IRREDUCIBLE VERTEX FUNCTIONS

- Self-energy and vertex function



❖ Spectrum: $A(\omega, \mathbf{k}) \rightarrow (\text{AR})\text{PES, STM, ...}$

❖ Energy: $E(T)/c_V(T) \rightarrow \text{Calorimetry}$



(2) The action of interacting fields is given by

$$\begin{aligned} S[\bar{\psi}, \psi] &= S_0[\bar{\psi}, \psi] + S_{\text{int}}[\bar{\psi}, \psi] \\ &= \int_0^\beta d\tau d\tau' \left\{ \bar{\psi}(\tau) \left(\frac{\partial}{\partial \tau} + \xi_k \right) \psi(\tau') - V[\bar{\psi}, \psi] \right\} \end{aligned}$$

The generating functional $Z[\bar{\eta}, \eta]$ is given by

$$Z[\bar{\eta}, \eta] = \int D[\bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi] + \int_0^\beta d\tau (\bar{\eta}(\tau) \psi(\tau) + \bar{\psi}(\tau) \eta(\tau))}$$

The generating functional $W[\bar{\eta}, \eta]$ is given by

$$W[\bar{\eta}, \eta] = \ln Z[\bar{\eta}, \eta]$$

(3) Introducing the conjugate function of $W[\bar{\eta}, \eta]$ as

$$\begin{aligned} \frac{\delta W[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau)} &= \langle \psi_k(\tau) \rangle_{\bar{\eta}, \eta} \equiv \phi(\tau) \\ \frac{\delta W[\bar{\eta}, \eta]}{\delta \eta(\tau')} &= -\langle \bar{\psi}_k(\tau') \rangle_{\bar{\eta}, \eta} \equiv -\bar{\phi}(\tau') \end{aligned}$$

The Legendre transformation of $W[\bar{\eta}, \eta]$ with $\eta(\tau)$ and $\phi(\tau)$ gives

$$\Gamma[\bar{\phi}, \phi] = \int_0^\beta d\tau (\bar{\eta}(\tau) \phi(\tau) + \bar{\phi}(\tau) \eta(\tau)) - W[\bar{\eta}, \eta]$$

The generating functional $\Gamma[\bar{\phi}, \phi]$ generates vertices functions and is called the effective action. The vertex functions $\Gamma^{(2n)}$ can be obtained by

$$\Gamma^{(2n)}(\tau_1, \dots, \tau'_1, \dots) = \left. \frac{\delta^{2n} \Gamma[\bar{\phi}, \phi]}{\delta \phi(\tau_n) \dots \delta \phi(\tau_1) \delta \bar{\phi}(\tau'_n) \dots \delta \bar{\phi}(\tau'_1)} \right|_{\bar{\phi}=\phi=0}$$

EXAMPLES:

1. The vertex function $\Gamma^{(2)}$

$$\Gamma^{(2)}(\tau, \tau') = \left. \frac{\delta^2 \Gamma[\bar{\phi}, \phi]}{\delta \phi(\tau') \delta \bar{\phi}(\tau)} \right|_{\bar{\phi}=\phi=0} = \frac{\delta}{\delta \phi(\tau')} \frac{\delta \Gamma[\bar{\phi}, \phi]}{\delta \bar{\phi}(\tau')}$$

Since

$$\begin{aligned} \frac{\delta \Gamma[\bar{\phi}, \phi]}{\delta \bar{\phi}(\tau)} &= \int_0^\beta d\tau' \left(\frac{\delta \bar{\eta}(\tau')}{\delta \bar{\phi}(\tau)} \phi(\tau') + \frac{\delta \bar{\phi}(\tau')}{\delta \bar{\phi}(\tau)} \eta(\tau) - \frac{\delta \eta(\tau')}{\delta \bar{\phi}(\tau)} \bar{\phi}(\tau') \right) \\ &\quad - \int_0^\beta d\tau' \left(\frac{\delta \eta(\tau')}{\delta \bar{\phi}(\tau)} \frac{\delta W[\bar{\eta}, \eta]}{\delta \eta(\tau')} + \frac{\delta \bar{\eta}(\tau')}{\delta \bar{\phi}(\tau)} \frac{\delta W[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau')} \right) \\ &= \int_0^\beta d\tau' \left(\frac{\delta \bar{\eta}(\tau')}{\delta \bar{\phi}(\tau)} \phi(\tau') + \eta(\tau) \delta(\tau - \tau') - \frac{\delta \eta(\tau')}{\delta \bar{\phi}(\tau)} \bar{\phi}(\tau') \right) \\ &\quad - \int_0^\beta d\tau' \left(-\frac{\delta \eta(\tau')}{\delta \bar{\phi}(\tau)} \bar{\phi}(\tau') + \frac{\delta \bar{\eta}(\tau')}{\delta \bar{\phi}(\tau)} \phi(\tau') \right) \\ &= \eta(\tau) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\delta \Gamma[\bar{\phi}, \phi]}{\delta \phi(\tau)} &= -\bar{\eta}(\tau) \\ \Gamma^{(2)}(\tau, \tau') &= \frac{\delta}{\delta \phi(\tau')} \eta(\tau) \end{aligned}$$

Consider the convolution of $\Gamma^{(2)}$ and $G_c^{(2)}$

$$\begin{aligned} &\int d\tau' \Gamma^{(2)}(\tau, \tau') G_c^{(2)}(\tau', \tau'') \\ &= \int d\tau' \left. \frac{\delta^2 \Gamma[\bar{\phi}, \phi]}{\delta \phi(\tau') \delta \bar{\phi}(\tau)} \right|_{\bar{\phi}=\phi=0} \left. \frac{\delta^2 W[\bar{\eta}, \eta]}{\delta \eta(\tau') \delta \bar{\eta}(\tau'')} \right|_{\bar{\eta}=\eta=0} \\ &= \int d\tau' \frac{\delta \eta(\tau)}{\delta \phi(\tau')} \frac{\delta \phi(\tau')}{\delta \eta(\tau'')} \\ &= \frac{\delta \eta(\tau)}{\delta \eta(\tau'')} \\ &= \delta(\tau - \tau'') \end{aligned}$$

The Fourier transform gives

$$\begin{aligned} \Gamma^{(2)}(k, \omega) G_c(k, \omega) &= 1 \\ \Rightarrow \Gamma^{(2)}(k, \omega) &= \frac{1}{G_c(k, \omega)} \end{aligned}$$

From Dyson equation

$$\frac{1}{G_c(k, \omega)} = \frac{1}{G_0(k, \omega)} - \Sigma(\omega)$$

$$\Rightarrow \Gamma^{(2)}(k, \omega) = \frac{1}{G_0(k, \omega)} - \Sigma(\omega)$$

2. The vertex function $\Gamma_0^{(2)}$ of free fields

The functional $W_0[\bar{\eta}, \eta]$ of free fields is given by

$$W_0[\bar{\eta}, \eta] = \ln Z_0[\bar{\eta}, \eta] = \int d\tau d\tau' \bar{\eta}(\tau) G_0 \eta(\tau')$$

The conjugate function of $W[\bar{\eta}, \eta]$ are

$$\frac{\delta W[\bar{\eta}, \eta]}{\delta \bar{\eta}(\tau)} = \phi(\tau) = \int d\tau' G_0 \eta(\tau')$$

$$\frac{\delta W[\bar{\eta}, \eta]}{\delta \eta(\tau')} = -\bar{\phi}(\tau') = \int d\tau \bar{\eta}(\tau) G_0$$

$$\Rightarrow W_0[\bar{\eta}, \eta] = \int d\tau \bar{\eta}(\tau) \phi(\tau) = \int d\tau d\tau' \bar{\eta}(\tau) \delta(\tau - \tau') \phi(\tau')$$

Since $\left(\frac{\partial}{\partial \tau} + \xi_k\right) G_0 = \delta(\tau - \tau')$

$$W_0[\bar{\eta}, \eta] = \int d\tau d\tau' \bar{\eta}(\tau) \left(\frac{\partial}{\partial \tau'} + \xi_k\right) G_0 \phi(\tau')$$

$$= - \int d\tau' \bar{\phi}(\tau') \left(\frac{\partial}{\partial \tau'} + \xi_k\right) \phi(\tau')$$

$$\Rightarrow \Gamma_0[\bar{\phi}, \phi] = -W_0[\bar{\eta}, \eta] = \int d\tau' \bar{\phi}(\tau') \left(\frac{\partial}{\partial \tau'} + \xi_k\right) \phi(\tau')$$

$$\Rightarrow \Gamma_0^{(2)}(\tau, \tau') = \frac{\delta^2 \Gamma[\bar{\phi}, \phi]}{\delta \phi(\tau') \delta \bar{\phi}(\tau)} \Big|_{\bar{\phi}=\phi=0}$$

$$= \left(\frac{\partial}{\partial \tau'} + \xi_k\right) \delta(\tau - \tau')$$

$$= G_0^{-1} \delta(\tau - \tau')$$

The Fourier transform gives

$$\Gamma_0^{(2)}(k, \omega) = \frac{1}{G_0(k, \omega)}$$

3. The vertex function $\Gamma^{(2)}$ is

$$\Gamma^{(2)}(k, \omega) = \Gamma_0^{(2)}(k, \omega) - \Sigma(\omega)$$

The vertex function $\Gamma^{(2)}$ contains one-particle irreducible (1PI)

vertices.